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Introduction to Intertemporal Equilibrium Theory: Indeterminacy, Bifurcations, and Stability[1]

Tapan Mitra

Department of Economics, Cornell University, 416 Uris Hall, Ithaca, New York 14853-7601
tml@cornell.edu

and

Kazuo Nishimura

Institute of Economic Research, Kyoto University, Yoshida Honmachi, Sakyoku, Kyoto 606, Japan
nishimura@kier.kyoto-u.ac.jp

This paper provides an overview of the basic concepts of intertemporal equilibrium theory, and discusses the frameworks and techniques used in this subject. It then goes on to introduce the main themes discussed in the papers included in this symposium issue. *Journal of Economic Literature* Classification Numbers: C61, D90, O41. © 2001 Academic Press

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1. EQUILIBRIUM OVER TIME: BASIC CONCEPTS, MODELS, AND METHODS

Over the past 25 years, there has been an enormous growth in the literature on the theory of intertemporal equilibrium and its applications. A basic ingredient of this theory is recognition of the fact that in making current investment decisions, which yield returns in the future, agents are necessarily "forward-looking" in the sense that their actions will be influenced by their beliefs regarding these future returns. These investment decisions could be regarding physical capital accumulation, the amount of education one acquires through schooling, the rate of extraction of mineral deposits, or the development of environmental resources for industrial purposes.

[1] We are grateful to Karl Shell for providing us with the opportunity to organize this symposium and for his advice and guidance. We thank Susan Schulze for her encouragement, assistance, and patience throughout this project. Some of the papers included in this symposium were first presented at a conference of Meiji-Gakuin University, Tokyo. We thank the organizers of the conference, especially Harutaka Takahashi, for providing the forum where the idea of organizing this symposium was first discussed. This symposium issue could not have been completed without the extraordinary cooperation we have received from the authors.

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The beliefs of agents regarding future returns on their investments can, of course, turn out to be incorrect. But it is plausible to proceed with the notion that beliefs that are at odds with the actual development of events cannot persist; any collection of agents' actions that we wish to call an "equilibrium" must validate (or at least not contradict) the beliefs on which they are based. Thus, one focuses on a notion of *equilibrium over time*, in which (i) given their beliefs, agents choose optimal actions according to their preferences, subject to the constraints they face; (ii) markets clear at each date; and (iii) the beliefs of the agents turn out to be correct.

This notion of equilibrium is studied principally in two types of intertemporal frameworks: (a) the overlapping generations model, in which agents are modeled as having finite lifetimes, which overlap with the lifetimes of some agents but not of others; (b) the infinitely lived agent model, where agents are modeled as dynasties, having no natural termination date to their "lifetimes." Clearly, the choice of the framework is determined to some extent by the particular issue one wishes to address.

For either framework, for the notion of equilibrium over time to be useful from the point of view of prediction of economic outcomes, it should be *determinate*. It would be convenient if, for example, there were a unique equilibrium path for each initial condition. But robust examples of nonuniqueness of equilibrium arise even in the atemporal framework. In the latter context, we do know that, in a generic sense, equilibria are locally unique.[2] So what one can hope for in the intertemporal context is that its equilibria are determinate in this sense.

In the context of the overlapping generations model, it is known that even this limited determinacy is not assured; there exist robust examples (in both the pure-exchange and the production settings) with a continuum of equilibria.[3] Thus, in examining issues for which the overlapping-generations model is the natural setting (for example, equilibrium bequest behavior), the study of determinacy of equilibria becomes especially important.[4]

The infinitely lived representative agent model behaves, of course, like an optimal growth model, thereby ensuring uniqueness of equilibrium paths (under the standard neoclassical assumptions). More generally, with a finite number of infinitely lived heterogeneous agents, it is known that equilibria are locally unique for almost every distribution of endowments.[5]

[2] See Debreu [9] for the basic result of the subject.

[3] For the exchange economy context, see Kehoe and Levine [13].

[4] See the paper by Nourry and Venditti [20] in this symposium for an example of such a study. The role of separability of preferences in ensuring determinacy of equilibrium in the overlapping generations exchange economy is explored in detail in Geanakoplos and Polemarchakis [11] and Kehoe and Levine [12].

[5] This result is due to Kehoe *et al.* [14].

However, when the standard assumptions are violated (for example, when there is an externality in production), the possibility of *indeterminacy* of equilibrium arises.[6] Investigation of the situations under which such indeterminacy will in fact prevail becomes important as a possible explanation of significantly different trajectories of economic growth of "similar" countries.

Given a determinate equilibrium path, it would be useful to be able to describe how the entire equilibrium path varies in response to variation in economic parameters relating to some aspect of tastes or technology or some policy parameter controlled by a monetary or fiscal authority. Samuelson [22] used the term *comparative dynamics* to describe this kind of analysis.[7]

The period of rapid growth in the literature on intertemporal equilibrium theory coincided with major contributions to the mathematical literature on nonlinear dynamical systems. An aspect of the development of the theory of dynamical systems concerns itself with the change in the trajectories of the system in response to a change in a parameter describing the system. This is, broadly speaking, the subject matter of *sensitivity analysis*. The methods of *bifurcation analysis* (which may be viewed as a branch of sensitivity analysis) appear to be potentially useful in conducting comparative dynamic exercises, especially in identifying the parameter values for which a *qualitative change* occurs in the nature of the equilibrium dynamics.[8] However, in applying these methods, one should be aware of a difference of emphasis in the development of the mathematical literature from its economic counterpart.

In the study of dynamical systems, the law of motion of the system is treated as a primitive. In the context of intertemporal economics, this law of motion is derived from the equilibrium conditions mentioned above and therefore inherits its properties from *its* primitives, such as tastes and technology, the objectives of the agents, and so on. Thus, the mathematical results regarding the nature of trajectories of a dynamical system, under some assumptions about its law of motion, become applicable in the study of intertemporal economics only when those properties can be justified starting from assumptions regarding the economic primitives. It is often the case that the mathematical literature points in the right direction without

[6] The paper by Boldrin *et al.* [7] in this symposium demonstrates global indeterminacy when there is a production externality in the consumption good sector.

[7] In cases where the notion of equilibrium coincides with an optimum, one might alternately describe the exercise as studying the effect of a change in an economic parameter on the optimal value and policy functions. For instances of these exercises, see the papers by Mitra and Nishimura [18] and Scheinkman and Zariphopoulou [21] in this symposium.

[8] See the paper by Antinolfi *et al.* [1] in this symposium for an example of the use of bifurcation analysis.

providing exactly the required result, which has to be suitably devised, given the particular economic problem under study.[9]

An infinite-horizon equilibrium trajectory is often hard to describe and characterize. So, quite often, one characterizes instead a set of states, which could be described mathematically as a *minimal closed invariant set*. This set could consist of a single *stationary* or a *steady-state* equilibrium, but it could also consist of the states associated with, say, a two-period cycle, or even a more complicated set. The presumption is that this set is also an *attractor* of the relevant dynamical system: equilibrium paths starting from arbitrary initial states get "close" to this set "rather quickly," so that the behavior of equilibrium paths away from this attractor may really be viewed as *transitional dynamics*.[10] For this practice to be fully justified, one needs to establish a *stability* property of the invariant set and also a suitable result on the *speed of convergence* of equilibrium paths to the attractor.[11]

The procedure described in the above paragraph can be taken one step further. In the case where the invariant set under study is simply a steady-state, one can clearly conclude that we have indeterminacy of equilibria by verifying that the steady-state equilibrium is *locally* a *sink* of the dynamical system generated by its Ramsey–Euler equations. Then, Ramsey–Euler paths starting near the steady-state will converge to the steady-state, thereby verifying the transversality condition as well. Thus, *all* such paths will be bona fide equilibrium paths resulting in indeterminacy. However, the converse procedure of concluding *determinacy* by checking that a steady-state is locally a *source* of the dynamical system generated by the Ramsey–Euler equations is a short-cut one can take only at grave peril.[12]

The papers collected in this symposium issue are contributions to various aspects of intertemporal equilibrium theory, some of which we have touched on above. The diversity of economic issues addressed in these contributions is testament to both the richness of the concept of equilibrium over time and the maturity that research in this area has attained. We provide, in what follows, "thumbnail sketches" of the papers with the hope that the reader will be prompted to quickly move on to the papers themselves and appreciate their full flavor.

[9] See the paper by Mitra and Nishimura [18] in this symposium for an elaboration of this point.

[10] In the paper by Bala and Sorger [3] in this symposium, simulation methods are used to justify the focus of the study on stationary equilibria.

[11] In the paper by Bhattacharya and Majumdar [5] in this symposium, the stability and speed of convergence to a steady state is examined in a stochastic setting.

[12] For an elaboration of this point, see the paper by Benhabib *et al.* [4] in this symposium.

2. UNCERTAINTY OF TECHNOLOGICAL PROGRESS AND THE BUSINESS CYCLE

In the paper “Growth Cycles and Market Crashes,” Boldrin and Levine [6] study an activity analysis economy with constant returns to scale and zero profits. At each moment of time, capital is a fixed factor and is priced according to the expected discounted value of future rents, which are in the form of consumption produced directly or indirectly from that capital. They use activity analysis to model the existing state of knowledge as a set of currently available activities. This set of available activities grows over time due to technological progress. In general, to make use of new activities, it is necessary to introduce new kinds of capital. The authors distinguish between *research*, the process of introducing new technologies, and *development*, the process of improving old technologies.

Only when existing technologies can no longer be improved upon will a new type of technology be put in place to replace the old one. The new type of technology is immature at the start, but can be improved on with time.

Under uncertainty about the discovery of new technologies, the replacement of old technologies by new ones generates the dynamics. Boldrin and Levine refer to this as *endogenous growth*, since the adoption of new technologies depends crucially on the economic decisions made by firms (that is, by the zero-profit condition). They show that technological progress under uncertainty leads to asymmetric movements of stock market prices characterized by gradual rises during booms, punctuated by sharp declines, and thus asymmetric growth cycles with long recoveries and short recessions. In this model, neither the stock market fluctuations nor the recession that follows the stock market crash is “bad.” Both are part of the *first-best solution* to the problem of maximizing the present value of utility from consumption.

3. DESTABILIZING CONSEQUENCES OF AN ACTIVE MONETARY POLICY

In “The Perils of Taylor Rules,” Benhabib, Schmitt-Grohé, and Uribe [4] examine whether active monetary policy, in which the monetary authority responds to increases in the rate of inflation with more than a one-for-one increase in the nominal interest rate, is actually stabilizing.

The principal line of argument developed here is that if nominal rates are to be non-negative, then corresponding to a steady-state equilibrium at which monetary policy is active, there is at least one steady-state equilibrium in which monetary policy is passive. This, in turn, means that even when the steady state, in which inflation is at the central bank's target rate and the monetary authority is taking an active stance against inflation, is locally determinate, it can be globally indeterminate. A basic contribution of the paper is that it establishes scenarios, with plausible values of the relevant parameters, in which (an infinite number of) equilibrium paths, starting near the active steady state, converge to a limit cycle around that steady state, or to a different steady state, in which the rate of inflation is so low that the zero bound on nominal interest rates forces the monetary authority to react passively. The existing literature typically rules out such equilibrium trajectories by conjecturing (either explicitly or implicitly) that such paths would be explosive. Thus, the results of this paper call for reassessment of the literature focusing on the stabilizing influence of Taylor Rules.[13]

The main result of the paper requires, in contrast to the local analysis typically offered in the literature, a careful study of the *global dynamics* of equilibrium trajectories. The paper makes particularly innovative use of mathematical results on bifurcation theory for two-dimensional systems, to provide the appropriate global analysis. In particular, in this framework, equilibrium behavior can be characterized completely by using the Bogdanov–Takens and Kopell–Howard bifurcation theorems.

4. RETURNS TO SCALE AND THE ROBUSTNESS OF GROWTH DYNAMICS

In "Growth Dynamics and Returns to Scale: Bifurcation Analysis," Antinolfi, Keister, and Shell [1] examine how the dynamic behavior of an overlapping-generations model with production depends on the degree of returns to scale of the reproducible factors of production, capital, and a *technology* variable to be interpreted as the output of basic scientific research. In their framework, there are constant returns to scale in production to capital and labor so that competitive firms will not invest in research because they cannot internalize the return from this investment. Thus, investment in research is modeled as being publicly funded by taxing wage incomes.

In this setup, the well-known AK model of endogenous growth theory is seen to be a bifurcation point in the parameter (returns to scale of the reproducible factors) space, making the qualitative features of the global

[13] This literature has been largely inspired by John Taylor's seminal paper [26] on Federal Reserve policy.

dynamics of that model necessarily nonrobust to changes in this parameter.[14]

The mathematical difficulty of classifying this bifurcation arises from the fact that it occurs on the boundary of the state space. The bifurcation is transcritical; however, verification of this by using standard methods of bifurcation analysis is possible only in a special case.

5. INDETERMINATE AND CHAOTIC GROWTH UNDER PRODUCTION EXTERNALITIES

In the paper "Chaotic Equilibrium Dynamics in Endogenous Growth Models," Boldrin, Nishimura, Shigoka, and Yano [7] study a two-sector model of endogenous growth with externalities and examine the issues of global indeterminacy and chaotic growth of equilibrium paths in this context. In this model, there are constant returns to scale in the production of the capital good, which makes unbounded growth possible. The aggregate stock of capital creates a positive externality in the production of the consumption good, leading to overall increasing returns in that sector. This is responsible for the persistent oscillations in the growth rate.

The authors identify parametric conditions under which the equilibrium dynamics will be globally indeterminate. This is contrasted with the usual local indeterminacy results in the literature, where an infinite number of equilibrium paths, starting from the same initial condition near a steady state, converge asymptotically to it.

The model predicts that when the external effect is very strong, the growth rate will oscillate forever along cycles of long periodicity or even over a chaotic attractor. The authors provide a characterization of the conditions under which this occurs, and they illustrate the features of these endogenous growth cycles both analytically and by means of numerical simulations.

The follow-up paper by Mitra [17], titled "A Sufficient Condition for Topological Chaos with an Application to a Model of Endogenous Growth," is concerned with developing analytic methods that enable one to determine whether a given model will generate chaotic equilibrium dynamics.

The principal difficulty appears to be the analytical verification of the *transversality condition* for "candidate" paths, that is, for those paths satisfying Ramsey–Euler conditions. While this verification is always

[14] This finding formalizes the criticism of nonrobustness of the AK model made earlier by Solow [25].

required to establish the existence of equilibrium paths, the specific difficulty in this context is that it has to be checked even for paths which are asymptotically aperiodic, since one is establishing the existence of *chaotic* equilibrium paths. Clearly, simulation methods might provide information that points in the right direction but is not suitable for a rigorous verification.

The paper provides an easily verifiable sufficient condition for topological chaos for one-dimensional unimodal maps, which can be applied even when the well-known Li–Yorke [15] condition is violated. This condition is then applied to the particular context of the endogenous growth model studied by Boldrin, Nishimura, Shigoka, and Yano [7] to verify the existence of chaotic equilibrium paths in that model under suitable parametric conditions.

6. LOCAL EXTERNALITIES IN HUMAN CAPITAL ACCUMULATION

In "A Spatial–Temporal Model of Human Capital Accumulation," Bala and Sorger [3] explore the interplay between local externalities and global participation in labor and credit markets in explaining patterns of human capital accumulation.

An agent's decision to acquire education is often determined by a complex pattern of interaction with other agents, and to capture this local effect, the authors set up a spatial model. The dynamics of accumulation are captured in terms of an overlapping-generations structure so agents are identified in space–time coordinates.

The authors focus on stationary equilibria and show that social stratification can arise in such equilibria; that is, some agents choose not to get educated and remain as workers and others get educated and become managers. This stratification can be sustained in equilibria even when neighborhoods are overlapping. An important class of stationary equilibria (called *regular* stationary equilibria) is shown to be locally stable; in simulation studies, starting from a randomly chosen initial distribution of human capital stocks (and specific structure of local spillovers), the economy is seen to converge quickly to a regular stationary equilibrium.

The literature on the growth of socioeconomic classes has particularly emphasized the role of migration and credit constraints as major factors leading to spatial stratification.[15] In contrast, here, agents have fixed locations, and there are no credit market imperfections, yet stratification arises

[15] See, for example, the important contribution by Loury [16].

endogenously. Thus, in this framework, children from poor neighborhoods choose not to get educated because the low human capital of their neighbors reduces their ability to acquire it.

7. UNCERTAINTY AND IRREVERSIBLE INVESTMENT DECISIONS

In "Optimal Environmental Management in the Presence of Irreversibilities," Scheinkman and Zariphopoulou [21] examine a problem of optimal environmental management where an environment can be converted for an alternative use. The key ingredients of the problem are that conversion can occur in stages, but it is *irreversible*, and that the future benefit from each use is *uncertain*.

The authors show that the value function for the problem is the unique viscosity solution of a Hamilton–Jacobi–Bellman equation. For utility functions exhibiting constant relative risk aversion, the optimal action is shown to be a function of the fraction of the environment that has not been converted and the ratio of the benefits from the two uses of it. The optimal policy in this case is given by an *exercise boundary*, described by a map from the ratio of benefits to the fraction of the environment to be preserved.

The important qualitative property of the exercise boundary is the following. Conversion should not start until the flow of benefits from the alternate use exceeds the flow of benefits from the environmental use by a certain margin. This bias toward a more conservative policy reflects the presence of irreversibilities, together with the improved quality of information over time of future benefits. Thus, this paper formalizes and generalizes the notion of "quasi-option value," introduced by Arrow and Fisher [2].

A particularly intriguing comparative dynamic exercise in this framework is that an increase in the degree of risk aversion leads to a *less* conservative policy when much of the environment is still intact. This is seen as arising from the need of a more risk averse agent to increase the diversification of assets.

8. STABILITY OF THE STOCHASTIC STEADY STATE

In the paper "On a Class of Stable Random Dynamical Systems: Theory and Applications,'" Bhattacharya and Majumdar [5] study the asymptotic stability of the stochastic steady state (an invariant distribution) for a random dynamical system, described by a state space, a class of maps on that space, and a probability measure according to which the maps from this class are drawn. Such a dynamical system can represent, for instance, the law of motion of paths generated by descriptive or optimal growth models under uncertainty.

The authors show that when the associated Markov process satisfies a "splitting condition," the process converges to a unique invariant distribution, and the convergence is exponentially fast in Kolmogorov distance. When the state space is an interval and the class of maps is monotone increasing, the splitting condition is also seen to be "almost necessary" for the stability result. In particular, for models of optimal growth under uncertainty studied by Brock and Mirman [8], which produce monotone optimal policy functions, stability of the stochastic steady state is (almost) equivalent to the splitting condition.

The authors go on to apply their methods to a special case in which the random dynamical system is generated by two maps, one monotone increasing and the other unimodal. Under appropriate assumptions one can show that the stochastic process enters an invariant subinterval of the state space in finite time, and on this subinterval, both maps are monotone (one is monotone increasing, and the other is monotone decreasing).

9. ALTRUISM, BEQUESTS, AND DETERMINACY OF EQUILIBRIUM

In "Determinacy of Equilibrium in an Overlapping Generations Model with Heterogeneous Agents," Nourry and Venditti [20] study the local determinacy of equilibrium paths in an overlapping-generations model with production and with two types of consumers. Both types of consumers live for two periods; however, one type cares only about its own utility, while the other is altruistic and cares about the welfare of its offspring. The authors provide conditions under which altruistic consumers leave positive bequests to their children at the equilibrium steady state. Under those conditions, the altruistic consumers can be seen as behaving like infinitely lived agents.

The paper complements the analysis presented in Muller and Woodford [19], who consider a model where one type of consumer is finitely lived and the other is infinitely lived, and there are many consumption and capital goods. In that framework, when the set of infinitely lived agents is "small," there exist equilibrium steady states that are locally indeterminate.

Nourry and Venditti [20] restrict their study to the one capital good case. They show that when bequests are positive and stationary in equilibrium, the steady state is locally determinate for *any* positive proportion (p) of altruistic agents. Further, they provide conditions under which the steady-state capital–labor ratio is independent of this proportion, p. There is, however, a discontinuity at $p = 0$ since the stationary capital–labor ratio corresponds to a steady-state of a Diamond [10] economy with nonaltruistic agents.

10. DISCOUNTING AND PERSISTENCE OF FLUCTUATIONS

In the paper "Discounting and Long-Run Behavior: Global Bifurcation Analysis of a Family of Dynamical Systems," Mitra and Nishimura [18] investigate how the long-run behavior of optimal paths is influenced by the rate at which the future utilities are discounted.

The principal difficulty in conducting this sensitivity analysis is that the optimal policy function, which determines the law of motion of the relevant dynamical system, cannot typically be solved in closed form, and the exercise of dynamic optimization places only minimal restrictions on it.

The paper proceeds to develop the theory under two conditions. The first is called *history independence*, which ensures that long-run optimal behavior is the same independent of initial condition, thereby generalizing the turnpike property. The second is called *unique switching*, and it avoids, once optimal cycles have emerged, a reversion to a stable steady state when the discount factor is decreased. Together, the two conditions allow a complete analysis that enables one to obtain a precise bifurcation diagram even without knowing the form of the actual optimal policy function.

Two well-known examples in the literature, due to Sutherland [24] and Weitzman–Samuelson [23], are examined in detail. The two conditions mentioned above can be verified directly in these frameworks. An interesting feature of these examples is that we do not always observe period 2 cycles being "born" and gradually developing into cycles with larger magnitude as the discount factor falls. In some cases, period 2 cycles appear past the bifurcation point fully "grown-up." Thus, in these cases, we are likely to observe either approximately stationary behavior or large cycles.

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